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# On Stieltjes relations, Painlevé-IV hierarchy and complex monodromy

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**Abstract**

A generalization of the Stieltjes relations for the fourth Painlevé (PIV) transcendents and their higher analogues determined by dressing chains is proposed. It is proven that if a rational function from a certain class satisfies these relations it must be a solution of some higher PIV equation. The approach is based on the interpretation of the Stieltjes relations as local trivial monodromy conditions for certain Schrödinger equations in the complex domain. As a corollary a new class of Schrödinger operators with trivial monodromy is constructed in terms of the PIV transcendents.

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## 1. Introduction and formulation of the results

In 1885 Stieltjes [1,2] found the following remarkable interpretation of the zeros of Hermite polynomials $H_n(z)$:

$$H_n(z) = (-1)^n \mathrm{e}^{z^2} \frac{\mathrm{d}^n}{\mathrm{d}z^n} \mathrm{e}^{-z^2}.$$

Consider $n$ particles on the line interacting pairwise with a repulsive logarithmic potential in the harmonic field. Then the equilibrium of this system is exactly the set of zeros of $H_n(z)$. More precisely, the extremum condition for the function

$$U(z_1, \ldots, z_n) = \sum_{j=1}^{n} z_j^2 - \sum_{j<k}^{n} \ln(z_j - z_k)^2$$

which is the system of the *Stieltjes relations*

$$\sum_{j\neq k}^{n} (z_k - z_j)^{-1} - z_k = 0 \qquad k = 1, \ldots, n \tag{1}$$

determines exactly the roots of the equation $H_n(z) = 0$. Importantly, although all these roots are real the result is true in the complex domain as well, i.e. all the complex solutions of system (1) are actually real and coincide with the zeros of the Hermite polynomial $H_n(z)$. The proof is not difficult and based on the fact that $y = H_n(z)$ satisfies the following second-order linear differential equation:

$$y'' - 2zy' + 2ny = 0$$

and this actually determines $H_n(z)$ up to a constant multiplier (see, for example, Szego's classical book [3, pp 140–1]). The sign $'$ means the derivative with respect to $z$.

The aim of this paper is to show that these relations have a natural analogue for the fourth Painlevé (PIV) transcendents (and their higher analogues) and to explain how all this is related to the theory of Schrödinger operators with trivial monodromy in the complex domain.

To explain the relation of the Stieltjes result with the PIV equation

$$2ww'' = w'^2 + 3w^4 + 8zw^3 + 4(z^2 - a)w^2 + 2b \tag{2}$$

let us first recall the well known fact that the logarithmic derivative of the Hermite polynomial $w = -(\ln H_n(z))'$ satisfies PIV with special parameters $a = (n+1), b = 2n^2$ (see [4–6]). Notice that the zeros of Hermite polynomials are the simple poles of the corresponding rational solution $w$ of PIV, each of them has the residue $-1$. The second remark is that the Stieltjes relations (1) are equivalent to the fact that the function $f = -(z + w) = -z + (\ln H_n(z))'$ has no constant terms at the Laurent expansions at all the poles or, equivalently, that all the residues of the function $f^2 = ((\ln H_n(z))' - z)^2$ are zeros:

$$\operatorname{Res} f^2(z) = 0.$$

One can easily check that the left-hand side of the relation (1) is proportional to the corresponding constant term in the Laurent expansion of $f$ at the pole $z = z_k$.

The first simple observation is that in this form this relation holds for any solution of the PIV equation.

**Theorem 1.** *For any solution $w$ of the PIV equation the residues of the function $(z + w)^2$ are zero at all the poles of the solution $w$:*

$$\operatorname{Res} (z + w)^2 = 0. \tag{3}$$

Actually we will prove a more general result about the following system introduced by Shabat and the author in [7] in relation to the spectral theory of Schrödinger operators under the name *dressing chain*:

$$(f_i + f_{i+1})' = f_i^2 - f_{i+1}^2 + \alpha_i \qquad i = 1, 2, \ldots, N \tag{4}$$

where $N = 2p + 1$ is odd, $\alpha_1, \alpha_2, \ldots, \alpha_N$ are some constant parameters and we assume that $f_{N+1} = f_1$. In [7] it was shown that this system has many remarkable properties, in particular it passes the Kovalevskaya–Painlevé test.

When $N = 3$ and $\alpha = \sum_{i=1}^{N} \alpha_i = -2$ the dressing chain (4) is equivalent to PIV: the function $w = -(z + f_1)$ satisfies the PIV equation (2) with $a = \frac{1}{2}(\alpha_3 - \alpha_1), b = -\frac{1}{2}\alpha_2^2$ (see [7, 8]).

Similarly a dressing chain with any odd period $N \geqslant 3$ is equivalent to some nonlinear ordinary differential equation on $f = f_1$ of order $N$ (or $N - 1$ if we fix the sum of $f_i$ to be $\alpha z$). Slightly abusing the terminology we will call such equations the *higher PIV equations* which together form the *PIV hierarchy*. We should mention that an equivalent hierarchy has also been considered by Noumi and Yamada [9] who were not familiar with the theory of the dressing chain [7, 8].

**Theorem 2.** *For any meromorphic solution of the dressing chain (4) the function* $f = f_1$ *has poles of the first order with integer residues. At any pole* $z_0$ *with* $\mathrm{Res}_{z=z_0} f = m$ *the following generalized Stieltjes relations are satisfied:*

$$\mathrm{Res}_{z=z_0} f^2 = \mathrm{Res}_{z=z_0} f^4 = \cdots = \mathrm{Res}_{z=z_0} f^{2|m|} = 0. \tag{5}$$

*In particular, the residues of the square of any such function* $f$ *are zero at all the poles of* $f$*:*

$$\mathrm{Res}\, f^2(z) = 0. \tag{6}$$

The main question is how strong are these relations. We will show that at least for the rational solutions they are indeed very strong and can be used as their characteristic property.

More precisely, let us consider a class of rational functions of the form

$$f = \sum_{i=1}^{n} \frac{m_i}{z - z_i} + \nu - \mu z \tag{7}$$

where $m_i$ are some integers. It is easy to show (see [7] and section 2 below) that this is a general form of the rational solutions of the dressing chains. In particular, for the PIV equation (2) all the rational solutions have the form $w = -(f + z)$, where $f$ has a form (7) with $m_i = \pm 1$, $\nu = 0$ and $\mu = \pm 1$ or $-1/3$ (see [6]).

**Theorem 3.** *If a rational function* $f$ *from class (7) satisfies the generalized Stieltjes relations (5) then* $f$ *is a rational solution of some higher PIV equation.*

For a generic solution the residues $m_i = \pm 1$ (see section 2), so we only have the usual Stieltjes relations (6), or explicitly[1]

$$\sum_{j\neq k}^{n} \frac{m_j}{z_k - z_j} + \nu - \mu z_k = 0 \qquad k = 1, \ldots, n. \tag{8}$$

They can be represented as the extremum conditions for the function

$$V(z_1, \ldots, z_n) = \mu \sum_{j=1}^{n} m_j \left( z_j - \frac{\nu}{\mu} \right)^2 - \sum_{j<k}^{n} m_j m_k \ln(z_j - z_k)^2$$

which also has a 'physical' interpretation: it is the potential of the system of charged particles of charge $m_j$ with logarithmic pairwise interaction in an external harmonic field with charge $\mu$ centered at $\nu/\mu$. In the case when all $m_i = 1$, $\nu = 0$ and $\mu = -1$ (or, equivalently, if all $m_i = -1$, $\nu = 0$, $\mu = 1$) we have the Stieltjes system.

Actually we will describe all the solutions of system (8) explicitly as zeros of certain polynomials: Schur polynomials if $\mu = 0$ and the wronskians of Hermite polynomials if $\mu \neq 0$ (see section 4 below). The main idea is to interpret the Stieltjes relations as the trivial monodromy conditions for certain Schrödinger operators in the complex domain and then to use the known results about such operators [10, 11]. More precisely, let us consider such an operator

$$L = -\frac{\mathrm{d}^2}{\mathrm{d}z^2} + u(z)$$

with a potential $u$ which is meromorphic in the whole complex plane. We will say that the operator *L has trivial monodromy* if all the solutions of the corresponding Schrödinger equation

$$L\psi = -\psi'' + u\psi = \lambda\psi$$

[1] As I have learnt from V G Marikhin in this form (with $m_i = \pm 1$ and $\nu = 0$) the Stieltjes relations for the rational solutions of PIV equations have been written (under the name 'Coulomb gas equations') in the recent paper [20]. The question of how strong these relations are was not addressed in this paper.

are also meromorphic in the whole complex plane for *all* $\lambda$.

Duistermaat and Grünbaum were probably the first to consider the problem of the classification of all such operators. In their fundamental paper on bispectrality [10] they have solved this problem in the class of the rational potentials decaying at infinity. Oblomkov [11] recently generalized this result to the case when the potential has a quadratic growth at infinity. Gesztesy and Weikard investigated the case of the potentials given by elliptic functions [12].

It turns out that the Stieltjes relations (3) are exactly the local trivial monodromy conditions for the following new class of Schrödinger operators related to PIV transcendents. Let $w$ be any solution of PIV equation (2) (which is known to be meromorphic in the whole complex plane) and let us consider the Schrödinger operator $L$ with the potential

$$u = w' + (w + z)^2. \tag{9}$$

**Theorem 4.** *For any solution $w$ of the PIV equation (2) the Schrödinger operator $L$ with potential (9) has trivial monodromy in the complex plane. The same is true for operators with the potentials*

$$u = f' + f^2$$

*where $f = f_1$ for any meromorphic solution of some dressing chain.*

As a by-product we have the proof of the following result first established by Calogero [13] (see also [14]). The following system of *Calogero relations*:

$$2\sum_{j\neq k}^{n}(z_k - z_j)^{-3} - z_k = 0 \qquad k = 1, \ldots, n \tag{10}$$

describes the equilibriums in the well known Calogero–Moser model with the potential

$$V_{\mathrm{CM}}(z_1, \ldots, z_n) = \sum_{j=1}^{n} z_j^2 + 2\sum_{j<k}^{n}(z_j - z_k)^{-2}.$$

**Theorem 5 (Calogero).** *The Stieltjes relations (1) imply the Calogero relations (10).*

For the reals these relations are equivalent but in the complex domain Calogero relations have many more solutions differing from the zeros of Hermite polynomials (see sections 3 and 4 below). For a discussion of the hierarchy of the similar relations for some classical polynomials and Bessel functions we refer to [15].

## 2. Local analysis of the dressing chain and the proof of the Stieltjes relations for PIV transcendents

Let us prove theorem 2, theorem 1 will then follow. The local expansions of the solutions of the dressing chain (4) near a pole (which we assume without loss of generality to be zero) have the form (see [7]): $f_i = a_i z^{-1} + b_i + c_i z + \cdots$. Substitution of this form into system (4) gives an infinite system of equations for the coefficients. The first two equations are

$$-(a_i + a_{i+1}) = a_i^2 - a_{i+1}^2 \tag{11}$$

and

$$2a_i b_i - 2a_{i+1} b_{i+1} = 0. \tag{12}$$

The first equation means that the coefficients $a_1, a_2, \ldots, a_N$ determine a periodic trajectory of the 2–2 correspondence

$$(x + y)(x - y + 1) = 0. \tag{13}$$

**Lemma 1.** *Any periodic trajectory of the correspondence (13) of an odd period must be an integer and contain zero.*

The proof is elementary. Let us first show that such a trajectory must be an integer. This follows from the fact that the image of $x$ under the $N$th iteration of the 2-valued mapping (13) consists of points of the form $x + k$, $-x + l$ with some integer $k, l$ such that $k \equiv N (\mathrm{mod}\, 2)$, $l \equiv (N-1)(\mathrm{mod}\, 2)$. In particular, if $N$ is odd then $k \neq 0$ so for periodicity one can only have $-x + l = x$, which means that $2x = l$, so $x$ is an integer since $l$ is an even number.

Now let us consider the absolute value of $x : u = |x|$. Under the mapping (13) $u$ may either stay unchanged or move by one in either the positive or negative direction. Obviously for periodic trajectories we have an even number of the last movements. This means that for an odd period we have an odd number of changes of sign which only can be possible if at least one element of the trajectory vanishes. This completes the proof of the lemma.

**Remark.** Notice that we have actually proved more: it follows from the proof that any periodic trajectory of the period $N = 2p + 1$ consists of integers between $-p$ and $p$. The extreme examples are $-p, -p+1, -p+2, \ldots, -1, 0, 1, \ldots, p-1, p$ and $1, -1, 0, 0, 0, \ldots, 0$. Only for the last sequence one has a family of solutions depending on the maximal number of free parameters (see [7]).

Now from the lemma it follows that at any pole of the solution of (4) at least one of the coefficients $a_i$ must be zero. Relation (12) shows that the product $2a_i b_i = 2a_{i+1}b_{i+1}$ is independent of $i$ and because $a_i = 0$ for some $i$ this is zero for all $i$. This means that if $a_j \neq 0$ then the corresponding $b_j = 0$, and therefore $\operatorname{Res} f_i^2 = 0$ for all $i$. Continuing in a similar way one can prove that if $a_j = m$ then all the coefficients at the Laurent expansions of $f_j$ at even powers $z^{2k}$ are zeros for all $k = 0, 1, \ldots, |m| - 1$. This implies theorem 2 (and therefore theorem 1).

Theorem 1 can also be proven directly from the local analysis of the PIV equation. Indeed, substituting the general form of the pole expansion of the solution $w$

$$w = \alpha(z - z_0)^{-1} + \beta + \gamma(z - z_0) + \cdots$$

into PIV equation (2) one can easily derive that

$$\alpha = \pm 1 \tag{14}$$

and:

$$\beta = -z_0.$$

The last relation means that the constant term of a similar expansion for the function $f = w + z$ at this pole is zero:

$$\begin{aligned} w + z &= \alpha(z - z_0)^{-1} + (\beta + z_0) + (\gamma + 1)(z - z_0) + \cdots \\ &= \alpha(z - z_0)^{-1} + (\gamma + 1)(z - z_0) + \cdots. \end{aligned}$$

This gives a direct proof of the Stieltjes relations for the general solution of the PIV equation.

**Remark.** For the PIV equation there is a theorem saying that all solutions are meromorphic in the whole domain. We believe that the same is true for solutions of the dressing chains (i.e. for the whole PIV hierarchy) but the proof is still to be found.

## 3. Stieltjes and Calogero relations as trivial monodromy conditions

Now we are going to prove theorem 4 leaving the proof of theorem 3 for the next section. Let us consider the Schrödinger equation

$$-\varphi'' + u(z)\varphi = \lambda\varphi \tag{15}$$

with a meromorphic potential $u$ having poles only of second order. Near such a pole (which can be assumed for simplicity to be $z = 0$) the potential can be represented as a Laurent series:

$$u = \sum_{i=-2}^{\infty} c_i z^i.$$

Following the classical Frobenius approach (see e.g. [16]) one can look for solutions of the form

$$\varphi = z^{-\mu}\left(1 + \sum_{i=1}^{\infty} \xi_i z^i\right).$$

The corresponding $\mu$ must satisfy the characteristic equation $\mu(\mu + 1) = c_{-2}$, which means that equation (15) has a meromorphic solution only if the coefficient $c_{-2}$ at any pole has a very special form:

$$c_{-2} = m(m + 1) \qquad m \in \boldsymbol{Z}_+. \tag{16}$$

This condition is in fact not sufficient: the corresponding solution $\varphi$ may have a logarithmic term. A simple analysis shows (see e.g. [10]) that the logarithmic terms are absent for all $\lambda$ if and only if in addition to (16) all the first $m + 1$ odd coefficients at the Laurent expansion of the potential are vanishing:

$$c_{2k-1} = 0 \qquad k = 0, 1, \ldots, m. \tag{17}$$

The relation of this theory with the Stieltjes relations is explained by the following simple but important lemma.

**Lemma 2.** *Let $f$ be a meromorphic function having poles of the first order with integer residues. The Schrödinger operator $L$ with the potential $u = f' + f^2$ has trivial monodromy in the complex domain if and only if at any pole $z_0$ with $\mathrm{Res}_{z=z_0} f = m$ the following relations are satisfied:*

$$\mathrm{Res}_{z=z_0} f^2 = \mathrm{Res}_{z=z_0} f^4 = \cdots = \mathrm{Res}_{z=z_0} f^{2|m|} = 0.$$

The proof is straightforward: one can easily check by the substitution of $f = \frac{\pm m}{z-z_0} + \sum_{k=0} \alpha_k (z - z_0)^k$ with $m \in \boldsymbol{Z}_+$ into $u = f' + f^2$ that $c_{-2} = m(m \pm 1)$ and that the trivial monodromy conditions $c_{2k-1} = 0, k = 0, 1, \ldots, m - 1$ are equivalent to the vanishing of the coefficients $\alpha_{2k} = 0, k = 0, 1, \ldots, m - 1$. A remarkable fact is that an additional relation $c_{2m-1} = 0$, which should be checked for the negative residue $-m$, is then fulfilled automatically.

Combining this lemma with the first two theorems proved in the previous section we come to theorem 4.

Now let us explain how this implies the Calogero result. Consider the rational function $w$ of the form:

$$w = \sum_{i=1}^{n} \frac{-1}{z - z_i}.$$

As we have shown, the Stieltjes relations imply the local trivial monodromy conditions for the potential $u = w' + (w + z)^2$. The function $u$ has a form

$$u = z^2 + \sum_{i=1}^{n} \frac{2}{(z - z_i)^2} \tag{18}$$

since the residues $c_{-1}$ must be zero (see (17)). It is easy to check that the second of the local trivial monodromy conditions $c_1 = 0$ for such a potential are precisely the Calogero relations. This proves theorem 5.

As we will see in the next section the Calogero relations holds not only for the zeros of the Hermite polynomials $H_k$ but also for the zeros of all their wronskians $W(H_{k_1}, H_{k_2}, \dots, H_{k_n})$. This means that in the complex domain the Stieltjes relations are actually much stronger than the Calogero relations.

## 4. Stieltjes relations and the rational solutions of the dressing chains

Consider now a rational function of the form (7)

$$f = \sum_{i=1}^{n} \frac{m_i}{z - z_i} + \nu - \mu z$$

where all $m_i$ are integers. We are going to describe all the functions of this form which satisfy the generalized Stieltjes relations (5). As a corollary we will prove theorem 3 which claims that such a function must be a rational solution of some higher PIV equation (4).

Notice first that the scaling transformations $f(z) \to \beta f(\beta z - \alpha)$ preserve both the class of functions (7) and the PIV hierarchy (4). Modulo these transformations, we have essentially only two different cases: $\mu = 0$ (with an arbitrary $\nu$) and $\mu = 1, \nu = 0$. Following the main idea of the previous section let us consider the Schrödinger operator $L$ with the potential $u = f' + f^2$. Lemma 2 says that the generalized Stieltjes relations (5) implies that the operator $L$ has trivial monodromy in the whole complex plane. Due to the relation $\mathrm{Res} f^2 = 0$ all the residues of $u$ are zero so the potential has a form $u = \sum_{i=1}^{n} \frac{2}{(z-z_i)^2} + c_0$ if $\mu = 0$ or $u = z^2 + c_0 + \sum_{i=1}^{n} \frac{2}{(z-z_i)^2}$ if $\mu = 1$.

Now we can use the results of Duistermaat–Grünbaum [10] and Oblomkov [11] which describe all such operators explicitly in terms of Darboux transformations.

Let us consider first the case $\mu = 0$. Following Adler–Moser [17] let us define the sequence of polynomials determined by the recurrence relation $P_k(z)'' = P_{k-1}(z)$ with $P_1 = z$

$$P_1 = z \qquad P_2 = \tfrac{1}{6} z^3 + \tau_1 \qquad P_3 = \tfrac{1}{120} z^5 + \tfrac{1}{2}\tau_1 z^2 + \tau_2, \dots$$

and let $W_n = W(P_1, P_2, \dots, P_n)$ be the Wronskian of these polynomials, which is also a polynomial in $z$ depending on $n$ additional parameters $\tau_1, \tau_2, \dots, \tau_n$. This is a special case of Schur polynomials also known as Burchnall–Chaundy (or Adler–Moser) polynomials. Duistermaat and Grünbaum [10] have proved that if a rational potential $u$ decays at infinity and satisfies all the local trivial monodromy conditions (16), (17) then (up to a shift $z \to z - a$) it must be equal to the second logarithmic derivative of such a polynomial $W_n$ with some parameters $\tau_1, \tau_2, \dots, \tau_n$:

$$u = -2(\log W_n(z))''.$$

The corresponding function $f$ is a rational solution of the Riccati equation $f' + f^2 = u$. It is easy to show, using the results of [10], that $f$ must be of the form

$$f = \frac{\mathrm{d}}{\mathrm{d}z} \log \frac{W_{n\pm1}}{W_n} \tag{19}$$

if $\nu = 0$ and

$$f = \frac{\mathrm{d}}{\mathrm{d}z} \log \frac{\hat{W}_n}{W_n} \tag{20}$$

if $\nu \neq 0$. Here $\hat{W}_n = W(P_1, P_2, \ldots, P_n, \mathrm{e}^{\nu z})$ is the Wronskian of the functions $P_1, P_2, \ldots, P_n, \mathrm{e}^{\nu z}$.

In the case $\mu = 1$, $\nu = 0$ we can use Oblomkov's theorem which says that any Schrödinger operator with trivial monodromy, and with the rational potential growing at infinity as $z^2$, has the form

$$L = -\frac{\mathrm{d}^2}{\mathrm{d}z^2} - 2\frac{\mathrm{d}^2}{\mathrm{d}z^2} \log W(H_{k_1}, H_{k_2}, \ldots, H_{k_n}) + z^2 + \text{const}$$

where $H_k(z)$ is the $k$th Hermite polynomial and $k_1, k_2, \ldots, k_n$ is a sequence of different positive integers (see [11]). The corresponding functions $f$ have the form

$$f = \frac{\mathrm{d}}{\mathrm{d}z} \log \frac{W(H_{k_1}, H_{k_2}, \ldots, H_{k_n}, H_{k_{n+1}})}{W(H_{k_1}, H_{k_2}, \ldots, H_{k_n})} - z \tag{21}$$

where $k_1, k_2, \ldots, k_n, k_{n+1}$ are again some different positive integers.

The formulae (19)–(21) give a complete description (modulo natural affine transformations) of functions $f$ of the form (7) which satisfy the Stieltjes relations (5). Now we are ready to prove theorem 3. Indeed we have seen that for any such $f$ the corresponding Schrödinger operator $L$ is a result of some number $m$ of the rational Darboux transformations applied to the operator $L_0 = -\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \mu^2 z^2$. Reversing this procedure we come back to $L_0$, then by taking $f_0 = -\mu z$ we can do one more step which only shifts $L_0$ by a constant. Now we can apply our rational Darboux transformations to return to the initial operator $L$. Thus we have constructed a closed chain of the rational Darboux transformations of an odd length $N = 2m + 1$, which is equivalent to the rational solution of the dressing chain (4) with $f_1 = f$ (see [7]). Theorem 3 is proven.

**Remark.** To identify all $f$ of form (21) which satisfy some higher PIV of a *given* order is actually a non-trivial task. In this relation I would like to mention the paper by Noumi and Yamada [18] where the rational solutions for the ordinary PIV equation ($N = 3$) have been described in terms the Schur functions for the special Young diagrams.

## 5. Some open questions

We have seen that the rational solutions of the PIV hierarchy can be characterized as certain rational functions satisfying the generalized Stieltjes relations (see theorem 3). It is natural to conjecture that these relations also characterize the general solutions of a PIV hierarchy among all the meromorphic functions of a certain order (in the sense of Nevanlinna) with integer residues. Rod Halburd suggested recently some interesting ideas which may help to prove this. As an intermediate case one can consider the special solutions of PIV equations expressed in terms of Weber–Hermite functions (see e.g. [6]).

Another interesting question: what are the analogues of the Stieltjes relations for other Painlevé transcendents? For example, for the second Painlevé equation (PII)

$$y'' = 2y^3 + zy + a$$

one can easily see that the constant terms in the Laurent expansions at the poles of the solutions must be zero, so we have the relation

$$\operatorname{Res} y^2 = 0$$

which can be considered as such an analogue.

One more intriguing problem is to understand what is a proper multi-dimensional analogue of the Stieltjes relations in the theory of multi-dimensional Baker–Akhiezer functions recently developed in [19].

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